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Relativistic Thermodynamics

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ABSTRACT

Thermodynamics, when formulated as a dynamic action principle, allows a simple and effective integration into the General Theory of Gravitation.

I. Introduction

Extensions of Einstein's theory of gravitation were sought, and found, from 1916 onwards. This paper deals with the incorporation of thermodynamics into General Relativity, the relativistic extension of thermodynamics. We begin by declaring the aims and the limitations of this work.

The prototype of a unified field theory is the unification of the theories of Maxwell, Einstein and Dirac that is defined by the lagrangian. But 'electrodynamics' is a vast edifice that includes, besides the elementary interactions of photons and electrons, the whole field of electromagnetic interactions of matter, including most of solid state physics. The Maxwell-Einstein system does not encompass electromagnetism in this widest sense; it is principally a relativistic theory of elementary particles, and it is easy to understand why: Both theories are formulated in terms of action principles, all it takes is to add the actions of both theories. It is significant that all unified theories studied so far are based on action principles, and this even applies to some proposals for combining Thermodynamics with General Relativity (Taub 1954, Bardeen 1970, Schultz 1970). Modern astrophysics is not based on an action principle, and yet Einstein's action has been invoked many times, since Eddington's book of 1926. The theory presented here has been used (Frønsdal and Wilcox 2011a, 2011b) to model the dark matter distribution of the Galaxy.

It is evident that a unification of thermodynamics with General Relativity must begin by formulating thermodynamics in terms of an action principle. To lay the foundations of thermodynamics in terms of an action principle is anything but new, for it was the inspiration and guiding light of Gibbs' famous 300 page paper (Gibbs 1875).

The aim of this work is to give a formulation of thermodynamics in the framework of eulerian field theory, where the dynamical variables are densities, pressure, temperature and entropy, all fields over space and time, that allows for its integration into General Relativity. The solution proposed in this paper is restricted in some ways; applications include only adiabatic processes. It is a start.

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The potential benefits of formulating a fully fledged action principle for thermodynamics extend far beyond the present context; perhaps the principal advantage is that each successful application contributes constraints on the lagrangian that must be respected in subsequent applications. One should aim at a “lagrangian for hydrogen” and a “lagrangian for water” that would encompass all properties of those substances, in all their phases.

In the strictest sense thermodynamics is the study of equilibria, but of course we shall not be confined to that narrow context. In the first place, to make meaningful contact with General Relativity, we have to deal with localized thermodynamics, some times called Extended Thermodynamics, and Hydrodynamics. In the second place we shall have to deal with systems with several components and several phases, and with a general class of equations of state.

It is clear that the focus on an action principle limits the scope of the theory. The most severe limitation of theories based on a variational principle is that the flows have to be at least piecewise irrotational and determined locally by a velocity potential, although limited vorticity can be accommodated in the manner of Onsager’s theory (Onsager 1949).^{*} A specific lagrangian determines an adiabat and applies exclusively to adiabatic fluctuations. That this is useful is due to the fact that, as a rule, non-adiabatic changes take place on a very long time scale. Most treatments of continuous, non adiabatic changes assume that the time development proceeds through a sequence of adiabatic equilibria. In the case of the simplest systems the adiabats may be indexed by a single parameter, related to the entropy. A single expression, containing a free parameter, then applies to a family of systems interrelated by the exchange of heat. Recognition of this fact, together with the approximate additivity of lagrangians of composite systems, is a promising starting point for a study of the entropy and the non adiabatic processes of heterogeneous systems. That is; the action principle provides a convenient framework within which to study entropy, the most difficult concept in thermodynamics. (See Fronsdal 2011, Fronsdal and Parthak, 2011.)

I.1. Dynamical variables and thermodynamical potentials

The dynamical variables include the metric, the temperature, the pressure, the entropy, one or more densities, and one or more irrotational vector fields that represent flow velocities.

The laws of classical, equilibrium thermodynamics are expressed in terms of potentials U, F, H and G (Massieu 1869). Each is defined as a function of specific variables (the natural variables), namely

$$U(S, \mathcal{V}), \quad F(T, \mathcal{V}) \quad H(S, P), \quad G(P, T).$$

According to Gibbs, a thermodynamic system is completely defined by any one of these functions; an explicit expression for a potential in terms of its natural variables is called a fundamental relation. The four potentials are related by Legendre transformations and the fundamental relations for a given system are equivalent.

^{*} Points of vortices are omitted from the spacial manifold and the velocity potential is a section of a fiber bundle over the manifold.

I. 2. The energy for the simplest system

The most fundamental concept introduced by Gibbs is the representation of the states of a system in terms of surfaces in a real space with coordinates V, T, P and S . Gibbs based the entire theory on a principle of minimal “energy” and on a subsidiary principle of maximum entropy. The energy is subject to independent variations of 2 of the variables, the other 2 held fixed at arbitrary values, the four variables are *a priori* independent. The Gibbsian surfaces are 2-dimensional surfaces in four dimensions, determined by the variational equations. Physical configurations are points on this surface.

The action is thus a function of the four independent variables. To define the system one specifies the potentials and obtains 2 independent relations among the variables that in turn define the Gibbsian two dimensional surface.

The relations obtained by variation of the generic action must have universal validity and each of the following definitions of “the energy” satisfies that criterion,

$$E_1 = F(\mathcal{V}, T) + ST + \mathcal{V}P, \quad E_2 = H(P, S) - ST - \mathcal{V}P,$$

$$E_3 = G(T, P) + ST - \mathcal{V}P, \quad E_4 = U(\mathcal{V}, S) - ST + \mathcal{V}P.$$

According to Gibbs, in the description of with adiabatic processes the energy is to be varied with entropy and pressure held fixed; this suggests identifying the action with E_1 , then independent variation of \mathcal{V} and T give the standard relations

$$\frac{\partial F}{\partial T} + S = 0, \quad \frac{\partial F}{\partial \mathcal{V}} + P = 0.$$

The system is thus defined by the free energy and the values chosen for P and S . To encode the properties of the system in the expression for the free energy F , rather than U or G , is often the preferred choice, see for example Rowlinson (1970). It is a natural choice that, by fixing the entropy, focuses on adiabatic processes.

I.3. Localization

The local extrapolation of thermodynamics seeks to promote these relations to field equations that describe spatial but, in the first instance, time independent configurations. Until further notice we consider systems with one component and a single phase. The function F and the variable S are interpreted as specific densities, the variable V as a specific volume. The mass density is $\rho = 1/V$ and densities f, s are defined by *

$$f(\rho, T) = \rho F(\rho, T), \quad s = \rho S.$$

* The representation of the entropy density s as a product ρS is meant to imply that the specific entropy density S is independent of the dynamical variables. This is a specialization, to a wide class of systems, that is adopted throughout this paper. For a discussion of some simple systems that do not belong to this class see Fronsdal and Parthak 2011.

The energy density is

$$\mathcal{H} := \int_{\Sigma} h + VP, \quad h = \rho \vec{v}^2/2 + f(\rho, T) + sT, \quad s = \rho S, \quad (1)$$

with ρ, S, T and P treated as independent variables. The vector field \vec{v} is interpreted as a flow velocity. Variation of the field T gives

$$\frac{\partial f}{\partial T} + s = 0. \quad (2)$$

For a one component system, in the case that the specific entropy density is uniform, this is the usual polytropic condition. In the case of multi component systems it has a significance that by far exceeds what is usually recognized.

Local variation of ρ (variations that vanish at the boundary of the domain Σ), with S and P held fixed, gives

$$\frac{h}{\rho} + \rho \frac{\partial(f/\rho)}{\partial \rho} = (\text{constant}).$$

The local thermodynamic pressure is defined by

$$\rho^2 \frac{\partial(f/\rho)}{\partial \rho} - p = 0$$

and the last equation reduces to the hydrostatic condition.

$$\text{grad} \frac{h+p}{\rho} = 0.$$

A complimentary variation of the density, with the mass fixed but the volume not, gives the result that the thermodynamic pressure must coincide with the “external” pressure P at the boundary. In the interior the thermodynamic pressure is balanced by external forces.

I. 4. Dynamics

We are going to extend the action principle to incorporate the hydrodynamic equation of motion for the velocity field, as well as the equation of continuity. We begin by introducing the Poisson bracket; for two functionals f and g ,

$$\{f, g\} = \int d^3x \left(\frac{\partial f}{\partial v_i} \partial_i \frac{\partial g}{\partial \rho} - f, g \right).$$

Then the two fundamental equations of hydrodynamics take the form

$$\dot{\rho} = \{H, \rho\} = -\text{div}(\rho \vec{v})$$

and

$$\dot{\vec{v}} = \{H, \vec{v}\} = -\text{grad} \frac{\partial h}{\partial \rho}.$$

The first equation is the equation of continuity and the second equation will be related to the Bernoulli equation. The energy functional as defined by Eq.(1) is thus a fully fledged hamiltonian, although the temperature does not (yet) have a canonical conjugate field.

We propose going a step further, passing to a lagrangian formulation. As far as we know, this can be done only if the velocity field is irrotational, expressible as

$$\vec{v} = -\text{grad } \Phi.$$

The correct form of the kinetic part of the action was discovered by Fetter and Walecka (1980). We define an action and a lagrangian density by

$$A = \int dt d^3x \mathcal{L} + VP, \quad \mathcal{L} = \rho \dot{\Phi} - h, \quad h = \rho \vec{v}^2/2 + f(\rho, T) + sT. \quad (3)$$

This makes Φ canonically conjugate to the density ρ . The equations of hydrodynamics are now variational equations. Variation of Φ and ρ gives

$$\dot{\rho} = -\frac{\partial h}{\partial \Phi}, \quad \dot{\Phi} = \frac{\partial h}{\partial \rho}.$$

The first is the equation of continuity and the other is the integral form of Bernoulli's equation (see below).

The lagrangian of Fetter and Walecka served only for hydrodynamics, since the temperature had been eliminated by means of an equation of state. In Gibbs' thermodynamics the temperature as an independent dynamical variable. The variation of T gives as before

$$\frac{\partial f}{\partial T} + s = 0. \quad (4)$$

This 'polytropic relation' is to be regarded as a relation satisfied by the dynamical variables ρ and T that acquires substance once the (specific) entropy has been assigned.

The function $f + sT$ is a function of three independent variables and

$$\text{grad}(f + sT) = \frac{\partial(f + sT)}{\partial T} \text{grad } T + \frac{\partial(f + sT)}{\partial \rho} \text{grad } \rho + \frac{\partial(f + sT)}{\partial S} \text{grad } S.$$

The first term vanishes by the equation of motion and the third term vanishes if S is uniform, hence

$$\rho \text{grad} \frac{\partial(f + sT)}{\partial \rho} = \text{grad } p.$$

Consequently, "Newton's equation" for the acceleration (the Bernoulli equation) is obtained by variation of the density, is

$$\rho \frac{D\vec{v}}{Dt} = -\text{grad } p.$$

This thermodynamic action principle, based on the hydrodynamic lagrangian of Fetter and Walecka, allows for incorporation into the framework of General Relativity, as shall be shown in Part II of this paper. That is not the only advantage of this formulation of thermodynamics. The following sections give a very short account of some other applications and generalizations.

!5. 5. Non adiabatic processes

The lagrangian applies to adiabatic changes only. Relaxation is non adiabatic and irreversible. What makes it possible to talk about adiabatic dynamics is the fact (the assumption) that relaxation takes place on a much longer time scale. The variational equations determine the temperature as a function of the other variables, but there is no equation for the time derivative \dot{T} . Over short time intervals the temperature follows the variations of the density, over long time intervals it is determined by the heat equation. To go further we should incorporate the heat equation into our framework, which is difficult since this equation describes irreversible phenomena and because the process involves external agents.

There is an alternative, and that is to incorporate additional degrees of freedom into the theory. For example, cooling due to electromagnetic radiation can be made adiabatic by incorporating the electromagnetic fields and the associated dynamics. Under favorable circumstances the interaction can be incorporated into the entropy term. freezing the electromagnetic field results in a fixed value of the entropy.

Some insight can be obtained on a case by case basis, especially by a study of mixtures and phase transitions.

I. 6. Generalizations

Here we describe a fairly general situation that can be encompassed by the theory in the present stage of development.

Consider a system described by a number of independent densities, a number of independent, irrotational vector fields, temperature, entropy and external pressure, and confined to a domain Σ in space. We suppose that we have an expression for the free energy density $f(\rho_1, \rho_2, \dots; T)$ and propose the following lagrangian density

$$\mathcal{L} = \sum_j' \rho_j (\dot{\Phi}_j - \vec{v}_j^2/2 - \phi) - f(\rho_1, \rho_2, \dots, \rho_n; T) - T \sum_i^n \rho_i S_i - \frac{\hat{a}}{3} T^4. \quad (5)$$

We have included the newtonian potential ϕ , in recognition of the fact that the ρ 's are interpreted as mass densities, and the Stefan-Boltzmann pressure, reserving comments for Section III. The first sum is over a set of independent densities, one for each independently conserved density. A velocity potential Φ_j is canonically conjugate to each conserved density ρ_j . (Fronsdal 2011). The second sum is over all independent densities, independently conserved or not. The central problem is the determination of

the total free energy density. The most primitive guess and a good approximation in some cases is that a first approximation is obtained by summation over the components,

$$f(\rho_1, \rho_2, \dots; T) = \sum_i f_i(\rho_i, T) ? \quad (6)$$

The specific densities S_1, S_1, \dots are usually taken to be uniform; that is, spatially constants. The appearance of an arbitrary constant in any adiabatic lagrangian is expected, for it must be possible to add heat and this must be reflected in the equations of motion. But the appearance of n arbitrary constants (one for each component) requires that we specify the path that the system will follow in this n -dimensional “entropy space” . For any adiabatic system they are determined by the properties of the equilibrium configurations; in certain circumstances the Gibbs-Dalton hypothesis may serve to fix the ratios, although corrections are required to account for properties of real gases. For a system with chemical reactions, as in the case of the molecular dissociation of hydrogen, it was shown that the addition of heat to the compound system leads along a path along which $S_1 = S_2$ is fixed. With this assumption the Saha equation (Saha 19) may be derived by variation of the the lagrangian with respect to the densities. In the case of a saturated van der Waals gas, the path is defined by the Maxwell rule of equal areas, derived from the subsidiar principle of maximal entropy, with the result that $T(S_2 - S_1) = \epsilon$, the evaporation energy. In this case too, non adiabatic (but reversible) changes are taking place in which the entropy is changing and both cases provide some insight into the use and the concept of entropy.

I.7. 7. Example, hydrogen atmospheres

Hydrogen is present in most stars, the molecular form at low temperatures and the atomic form at higher temperatures. Here we consider the phenomenon of molecular dissociation, below the ionization temperature. The hamiltonian density, in a first approximation, is

$$h = \rho(\lambda + \vec{v}^2/2 + \phi) + \epsilon\rho_2 + \sum_i \mathcal{R}_i \rho_i \ln \frac{\rho}{k_i T^{n_i}}.$$

The index is 1 for the molecular component and 2 for the atomic component, ϵ is the molecular binding energy. The entropy is included as $S_i = -\mathcal{R}_i \ln k_i, i = 1, 2$. Variation of the densities, with $\rho = \rho_1 + \rho_2$ fixed, leads to the following equation for the concentration $r = \rho_2/\rho$ at fixed density

$$\frac{r^2}{1-r} = \frac{1}{e\rho} T^{.5} e^{-\epsilon/\mathcal{R}T}.$$

This is Saha’s equation. It was derived under the assumption that $S_1 = S_2$.

8. Phase transitions

For a pure substance some simple phase transitions and the associated dynamics is described by using the free energy of a van der Waals gas. As the temperature is lowered (non adiabatically) to the domain where the density is no longer determined by temperature and pressure the system behaves in some respects like a mixture, but surface tension and gravity leads to separation of the phases. The theory describes the equilibrium configurations and adiabatic dynamics but not the process whereby the system gives up or gains heat from its surroundings, nor the process of separation of the phases. In the saturated, separated phase Eq.(6) is valid. Variation of the densities gives the pressure and locates the critical point but it does not yield Maxwell's rule. Maxwell's rule can be derived from Gibbs complimentary variational principle of maximal entropy.

The question of the validity of Eq.(6) is vital in other contexts, such as the propagation of sound and the determination of critical parameters in gaseous mixtures. The latter problem is often formulated in terms of the free energy, while sound speeds are derived from very different considerations. This brings us to what is without a doubt the most radical consequence of adopting the action principle for all applications of thermodynamics: Every successful application gives information about the lagrangian. For example, the study of sound propagation in mixtures at normal temperatures suggests corrections to the formula (6) by interaction terms of the very specific form (Fronsdal 2011)

$$\alpha_{ij}(\rho_i\rho_j)^k,$$

The best account of critical parameters of mixtures have been obtained by a modification of the van der Waals formula for the pressure, but preliminary results based on the following formula for the free energy are very encouraging,

$$f(\rho_1, \rho_2, \dots; T) = \sum_i f_i(\rho_i, T) + \alpha_{ij}(\rho_i\rho_j)^k. \quad (6)$$

The particular form of the interacting term is suggested by observation of the excess free energy. When more data on sound propagation at low temperatures become available it will be imperative to reconcile observations of critical phenomena with sound speeds in mixtures, using the same expression for the free energy.

The ultimate goal of thermodynamic phenomenology should be to determine the lagrangian for each substance, including mixtures of all kinds and including all phases. and accounting for a large family of adiabatic properties.

II. Thermodynamics and General Relativity

Following Taub (1954), Schultz (1970) and Bardeen (1970), we propose a partial unification of thermodynamics and General Relativity. But unlike previous suggestions this one is based on, indeed defined by, a specific lagrangian density,

$$\mathcal{L}_{tot} = \frac{1}{8\pi G}(R + \Lambda) + \mathcal{L}_{matter},$$

where the second term is a relativistic extension of the lagrangian density in Eq.(5), *

$$\mathcal{L}_{matter} = \frac{1}{2} \sum_j' \rho_j (g^{\mu\nu} \psi_{j,\mu} \psi_{j,\nu} - c^2) - f(\rho_1, \rho_2, \dots, \rho_n; T) - T \sum_1^n \rho_i S_i + \frac{\hat{a}}{3} T^4.$$

The radiation term may perhaps be included in the free energy, but we prefer to add it explicitly. The first sum extends over a subset of 1...n and includes the densities that are separately conserved. The constant Λ is the cosmological constant and in the case of an astrophysical system that extends to infinity it is interpreted as the pressure at the boundary at infinity. It is a simple extension of the non relativistic lagrangian to which it reduces when in the kinetic term one expands

$$\psi_i = c^2 t + \Phi_i$$

and takes the limit as c tends to infinity. Of the dynamical metric there remains, to this order, only the Newtonian potential ϕ . Up to terms of order $1/c^2$, ,

$$g^{\mu\nu} \psi_{i,\mu} \psi_{i,\nu} - c^2 = c^{-2} (1 - 2\phi/c^2) (\dot{\psi}_i)^2 - (\text{grad } \psi_i)^2 - c^2 = 2\dot{\Phi}_i - (\text{grad } \Phi_i)^2 - 2\phi.$$

The action is relativistically invariant (invariant under diffeomorphisms) and it follows that the Bianchi identities are satisfied by virtue of the equations of motion.

The matter energy momentum tensor is

$$T_{\mu\nu} = \sum_i \psi_{i,\mu} \psi_{i,\nu} - \mathcal{L} g_{\mu\nu}.$$

Hence \mathcal{L} , on shell, is the pressure, just as it is in the non relativistic theory. The expression is similar to that of Tolman (1934), with some differences:

1. Any number of densities and vector fields can be accommodated.
2. The charges that are conserved in the non relativistic theory remain conserved in General Relativity,

$$\partial_\mu J_j^\mu = 0, \quad J_j^\mu = \sqrt{-g} g^{\mu\nu} \psi_{j,\nu}.$$

* An example of this action principle appears in Fronsda (2007).

3. The lagrangian is determined by the non relativistic free energy and the interpretation is the same as in the non relativistic theory. Chemical reactions and changes of phase can be described precisely as in non relativistic thermodynamics.

4. All the equations of ordinary thermodynamics and hydrodynamics are recovered in the non relativistic limit.

5. The simplest examples (ideal gas spheres, radiation neglected) lead to equations of motion that are almost identical to those used in simple stellar models.

6. The black body radiation pressure and energy is included by simply adding the term $T^4/3$ to the lagrangian density. This is appropriate in a region where radiation is believed to be in equilibrium with matter, as in the photosphere of the Sun. It raises or lowers the effective adiabatic index towards the asymptotic limit $n = 3$ as the temperature goes to infinity. It is an improvement of the approximation introduced by Eddington (1926) who postulated that the ration $p_{\text{gas}}/p_{\text{radiation}}$ is uniform.

7. The temperature is an independent field, fixed by the equation of motion, Eq.(2). This is important in the case that radiation is taken into account for this affects the polytropic relation. It leads to a correct inclusion of radiation pressure and energy and obviates the need to make Eddington's assumption.

8. As to the interpretation of the variables it should be noted that the pressure is the on-shell value of the lagrangian density, so that the term $-pg_{\mu\nu}$ in the energy momentum tensor is the same in both theories, but the dynamics is in the expression for the lagrangian and it is lost when the lagrangian is replaced by its value on shell. As to the other term, it can be compared to Tolman's expression only in the case of a simple system. There is no conserved energy density in General Relativity, but in the non relativistic limit

$$\rho\dot{\psi}\dot{\psi} = \dot{\psi}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = h + \mathcal{L},$$

and on shell it is $h + p$, where h is the hamiltonian. Hence Tolman's ρ corresponds to the hamiltonian density. An essential difference is that Tolman's formula deals with on shell quantities that obey no dynamics besides what is implied by the Bianchi identities.

9. The entropy enters in precisely the same way as in the classical theory. Any advance in the conceptual or operational meaning of entropy is transferred to the relativistic theory, without additional complications.

To all this must be added a discussion of the correct identification of mass density. The only effective definition of mass in General Relativity is that obtained in any empty region of space time, where the Schwarzschild metric can be directly observed. It is contained in the form of either g_{00} or g_{rr} . This definition can be usefully extended to regions of low matter density, thus for example

$$M(x) = \frac{r}{2}(c^2g^{00} - 1).$$

If the metric is determined locally, by the observation of orbital velocities, then this is the correct definition, for non relativistic dynamics is affected by g^{00} , much less by the radial component of the metric. Yet it is customary to define the mass density in terms of g_{rr} ,

in a manner that is inconsistent with the transformation law of a density. If there is a useful definition of mass density then it is the time component of the conserved current. There is no conserved current in Tolman’s approach; that theory abandons one of the key concepts of hydrodynamics. The integrated density is an invariant characteristic of a star, but it is not clear that the value of this quantity must agree with the mass as determined, locally or at infinity, by the Schwarzschild metric.

III. Example, dark matter

An interesting equation of state for stellar interiors, first proposed by Stoner() and Chandrasekhar(), is based on a degenerate Fermi gas. A variant was used in a celebrated paper by Oppenheimer and Volkov() and an account of it may be found in Landau and Lifschetz(). Another variant of this equation of state was used recently to model the dark matter in the Milky Way. (Fronsdal and Wilcox 1101a, 1101b)

Let f be the internal energy density. Little or no attention will be given to the temperature; it may be assumed, either, that it is constant, or that the entropy is zero; in either case f is principally a function of the density. Variation of the action with respect to the density gives, in the case of a spherically symmetric metric,

$$\frac{1}{2}(c^2 g^{00} - 1) =: \phi = \frac{df}{d\rho}.$$

Taking the gradient leads to the usual hydrostatic conditions relating $\rho \text{grad } \phi$ to the gradient of the pressure

$$p = \rho \frac{\partial f}{\partial \rho} - f.$$

That is, the gravitational potential is identified with the chemical potential. We note that

$$\rho = \frac{dp}{d\phi}.$$

Examples studied by Fronsdal and Wilcox (1011a) have (omitting constants) $\rho(\phi) = \sinh^4 \phi$. The non relativistic approximation leads to the the Laplace equation, $\Delta \phi = \rho$,

$$r^{-2} \partial_r r^2 \partial_r \phi = \sinh^4 \phi,$$

a “sinh-Emden equation” that is solved exactly by

$$\phi(r) = \ln(1 + b/r).$$

It can be used to account for the observed rotation curve of visible stars in the Milky Way. It predicts nearly constant orbital velocities up to the distance $r = b$. The solution is regular all the way to the center and reveals a small core that resembles a black hole. Further work is expected to produce an equation of state that can account for the high gravitational fields observed at about 1^{16} cm from the center. The presence of a non vanishing mass density appears to prevent the formation of black holes but to permit

very massive accumulations of dark matter. (The equations of state used in this example are not applicable to standard, visible matter.)

References

- Bardeen, J.M., A variational principle for rotating stars in General Relativity, *Astrophys. J.* **162**, 7 (1970).
- Chandrasekhar, S., *An Introduction to Stellar Structure*, U. Chicago Press 1938.
- Eddington, A.S., *The internal constitution of stars*, Dover, N.Y. 1959.
- Fetter, A.L. and Walecka, J.D., *Theoretical Mechanics of Particles and Continua*, MacGraw-Hill NY 1980.
- Fronsdal, C. , “ Thermodynamics of fluids, in progress. Ideas from this book appear in unpublished papers, “Heat and Gravitation. I. The action Principle”
arXiv 08124990v2 revised 2010 and “Heat and Gravitation. III. Mixtures” arXiv
- Fronsdal, C., Ideal Stars and General Relativity, *Gen.Rel.Grav.* **39** 1971-2000 (2007).
- Fronsdal, C. and Pathak, A., “On Entropy in Eulerian Thermodynamics”,
submitted for publication; arXiv
- Fronsdal C. and Wilcox, T.J. “An equation of state for the dark matter in the Milky way”, submitted for publication; arXiv
- Gibbs, J.W., “On the equilibrium of heterogeneous substances” *Trans.Comm.Acad.* 1878.
- Massieu, F., “Mmoire sur les fonctions catactristiques des divers fluides et sur la théorie des vapeurs”, *Comptes Rendues de l’ Acadmie des Sciences de L’Institut National de France*, 1876.
- Onsager, L., (1949).
- Rowlinson, Liquids and liquid mixtures”,
- Saha, M.N., “On a physical theory of stellar spectra”, *Proc. R.S. Series A*, **99**, 135-153 (1921).
- Schultz, B.F. Jr., “Perfect fluids in General Relativity,
Velocity potentials and a variational principle”, *Phys.Rev.D* **2**, 2762-2771 (1970).
- Taub, A.H., “General relativistic variational principle for perfect fluids”,
Phys.Rev. **94**, 1468 (1954).
- Tolman, R.C., *Relativity, Thermodynamics and Cosmology*, Clarendon, Oxford 1934.